Gradient Technique and the Maximum Principle

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The equivalence of the Lagrange multiplier equations and the adjoint equations of the maximum principle or the multiplier rule of the calculus of variations is well known (*1-3*). This fact is also pointed out in subject paper (*4*). The basic difference between the gradient technique approach and the maximum principle lies in the fact that the generally used maximum principle approach uses the optimum control variable while the gradient technique uses any nominal control variable (*2*).

From a computational standpoint, the generally used discrete maximum principle approach solves a two-point boundary value problem in difference equations while the gradient approach minimizes or maximizes the performance index directly. The former approach is known as approximation to the problem or approximation in function space and the latter is known as approximation to the solution or approximation in policy space (*2, 3, 5, 6*). These different approaches have been discussed in detail in the literature (*6*). The advantage of the gradient approach lies in the fact that the convergence is monotone convergence before the optimal solution is reached and avoids the stability problem in solving boundary value problems. Furthermore, in general, a better solution is obtained after each iteration. Notice that in the approximation to the problem, each iteration obtains an unwanted or wrong solution (*6*) in the sense that the boundary conditions are not satisfied. The correct solution is obtained only if the problem converges. For a very complicated industrial process, this difference is very important. This is because in practice the true optimal solution can never be reached for a complex industrial process. The results of any iteration in the gradient approach can be used in actual production.

The gradient technique has important disadvantages (*2, 6*). One is the slow convergence rate. This is especially true as the optimum is approached. Thus, accurate solution to the optimum is very difficult to obtain. Because of this difficulty, the gradient technique should be used only for very complicated problems which cannot be solved easily by other techniques. The gradient technique can also be used to obtain better initial approximations for highly unstable problems.

It is well known that discrete maximum principle can solve complex multistage processes (*7*). However, both maximum principle and dynamic programming are not suited for large nonlinear complex processes. In a complex chemical plant, such as a modern refinery, the basic serial structure is completely disappeared and even a homogeneous serial arrangement of only three stages is seldom seen. It should be noted that the example in (*4*) still has the basic serial structure. The discussion in (*4*) concerns the degree of difficulty in solving very complex problems. For almost all practical problems such as an operating chemical plant or a refinery, the true optimum cannot be obtained, for all practical purposes, by almost any existing technique including the maximum principle. However, better operating conditions can be obtained by the gradient technique. In fact, each iteration forms an improvement over the previous iteration. Furthermore, because of the constantly fluctuating characteristics, the data of the process are changing constantly.

However, for a reasonably simple complex process, such as the example in (*4*), several of the existing optimization techniques can be used. Since we know the gradient technique converges slowly and also since the problem is reasonably simple and is probably stable when solved as a boundary value problem with reasonable initial approximations, maximum principle is probably the better approach for this particular example. It should be noted that this particular example is only used for illustration purposes.

In the above discussion, the maximum principle method is used in a fairly narrow sense in that the usual numerical method is used to solve the system of equations resulting from the application of maximum principle. When the analytical solution cannot be obtained, the maximum principle only provides a system of equations. The numerical solution of this system of nonlinear equations is not simple. Many numerical methods have been developed based on this system of basic equations. In fact the gradient technique was developed to solve this system of equations resulting from the application of maximum principle or calculus of variations (*5, 6*). Notice that the basic equations resulting from the maximum principle or calculus of variations form nonlinear boundary value problems for complex nonlinear problems. These boundary value problems cannot be solved easily by the traditional maximum principle or calculus of variations approach due to stability problem. Thus, most of the numerical techniques for solving dynamic optimization problems such as the gradient technique are essentially techniques for solving the maximum principle equations. An important exception in this respect is the dynamic programming algorithm.

LITERATURE CITED

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